Notes on pandigital squares in base n

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By a pandigital number I mean pandigital numbers without repeating digits that don't start with a zero.

Given any n digit number p in base b with a_i as the digits it can be written as:

$$\begin{aligned} p &= a_{n-1}b^{n-1} + a_{n-2}b^{n-2} + ... + a_0 \\ &= \sum_{i=0}^{n-1} a_i b^i \\ &= \sum_{i=0}^{n-1} \left(a_i + a_i \left(b^i - 1\right)\right) \\ &= \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} a_i \left(b^i - 1\right) \end{aligned}$$

Representing the n digit repunit in base b as $R_n^{(b)}$

$$p = \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} a_i \left(b - 1\right) R_i^{(b)}$$

When p is a pandigital number in base b there are b digits $0...\left(b-1\right)$, so $n = b$.

Also, in a pandigital number $\sum_{i=0}^{n} a_i$ is the sum of its digits and a simple arithmetic series.

$$p = \frac{n\left(n-1\right)}{2} + \sum_{i=0}^{n-1} a_i \left(n - 1\right) R_i^{(n)}$$

When n is even

$$p = (n-1)\left(\frac{n}{2} + \sum_{i=0}^{n-1} a_i R_i^{(n)}\right)$$

So $n-1$ is always a factor of even pandigital numbers. This is unsurprising e.g. in ten digit pandigitals the digits add up to a multiple of nine, so the whole number is divisible by nine.

When n is odd it makes more sense to extract $\frac{n-1}{2}$ as this is always an integer

$$\begin{aligned} p &= \frac{n(n-1)}{2} + \sum_{i=0}^{n-1} a_i (n-1) R_i^{(n)} \\ &= \frac{n-1}{2}\left(n + 2\sum_{i=0}^{n-1} a_i R_i^{(n)}\right) \end{aligned}$$

So $\dfrac{n-1}{2}$ is always a factor of odd pandigital numbers

$n + 2\sum_{i=0}^{n-1} a_i R_i^{(n)}$ is also a factor of p and must be odd.

If $\dfrac{n-1}{2}$ contains an odd number of 2s as prime factors then p cannot be square as $\left(n + 2\sum_{i=0}^{n-1} a_i R_i^{(n)}\right)$ will not have a 2 as a prime factor.

Considering some values of n and applying the observation that odd pandigitals with an odd number of 2s as prime factors in $\dfrac{n-1}{2}$ tells us some bases that have no pandigitial squares.

n	factors of p	square p possible
4	3	yes
5	2	no
6	5	yes
7	3	yes
8	7	yes
9	2^2	yes
10	3^2	yes
11	5	yes
12	11	yes
13	$2, 3$	no
14	13	yes
15	7	yes
16	$3, 5$	yes
17	2^3	no
18	17	yes
19	3^2	yes
20	19	yes
21	$2, 5$	no

In general, if the base can be written in the form $2^c d + 1$ where c is even and d is odd then there will be no be no pandigital squares.
I can also use this to optimize the search for more pandigital squares by only checking roots with particular factors.